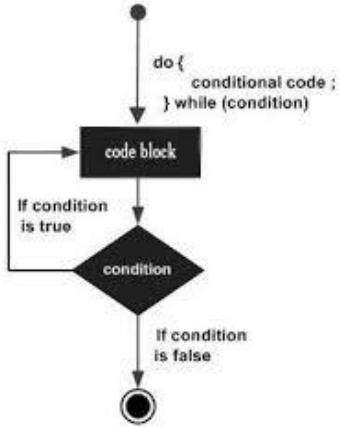
Lecture 10

Greedy Algorithms I



Announcements

- Homework 4 due Sunday night
- Group Meetings continue
- Quiz 1 due next Friday
  - Released earlier next week
  - A few questions from each topic

Greedy Algorithms

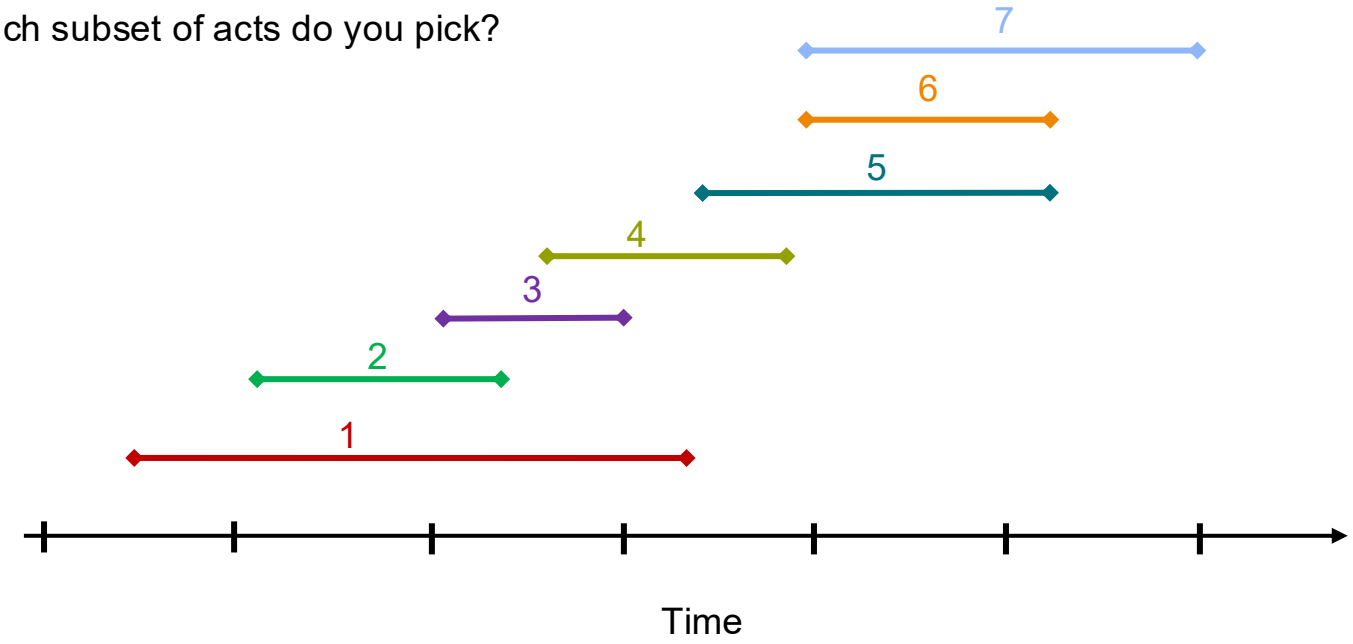
We are moving on to our study of algorithm design

- **Greedy**
- Divide-and-conquer
- Dynamic Programming
- Network Flow

Interval Scheduling

Imagine you are going to a music festival and want to see as many different acts as possible

- Which subset of acts do you pick?



Interval Scheduling

Let write this problem more generally

- Acts $1, 2, \ldots, n$
- $s(i)$: start time of act i
- $f(i)$: finish time of act i
- Acts i, j are **compatible** if they do not overlap
- Set A of acts is **compatible** if all pairs in A are compatible
- Set A of acts is **optimal** if it is compatible and no other compatible set is larger

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Goal: find optimal set of acts

Greedy Algorithms

Main idea is to make one choice at a time

- Roughly: choose the thing that looks best and never look back
- We will sort arts in some "natural order" and choose acts one by one if they are compatible with the acts already chosen

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Algorithm sketch:

$R \leftarrow$ set of all shows sorted by some property
$A \leftarrow \emptyset$
while R is not empty **do**
    take first show i from R
    add i to A
    delete i and all overlapping acts from R

Exercise I

Q: Suppose an algorithm includes a step that sorts a list of n items. Then its running time is:

a) $O(n \log n)$

b) $\Theta(n \log n)$

c) $\Omega(n \log n)$

d) None of the above

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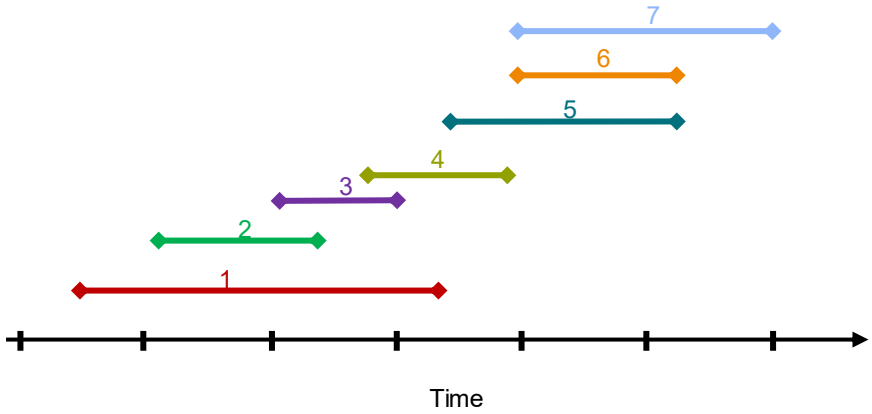
We can't provide an upper bound, because we don't know everything the algorithm is doing; however, we can provide a lower bound because MergeSort is $\Theta(n \log n)$

"Natural Order"

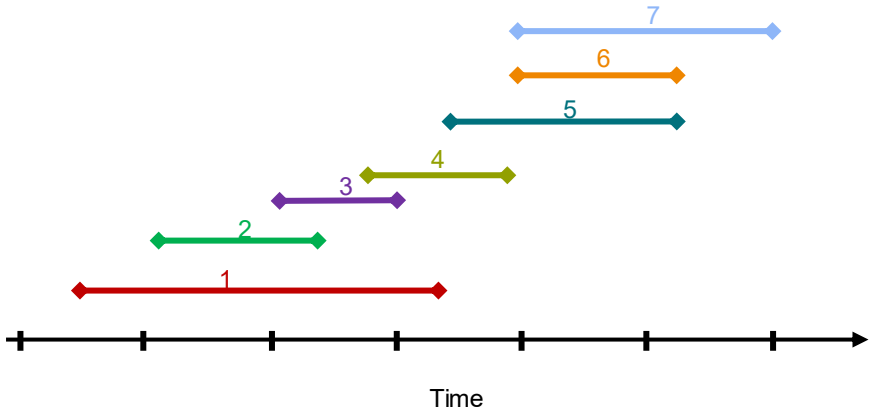
Start time: consider acts in ascending order of $s(i)$



"Natural Order"

Start time: consider acts in ascending order of $s(i)$

- Not optimal in this example



"Natural Order"

Start time: consider acts in ascending order of $s(i)$

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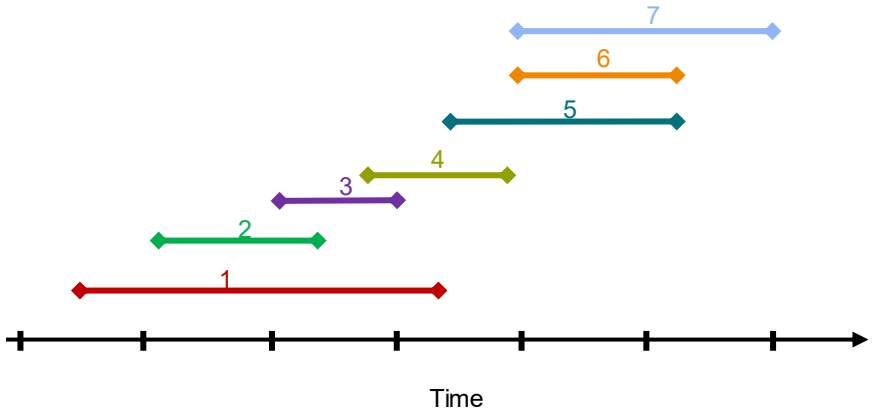
Shortest time: consider acts in ascending order of $f(i) - s(i)$



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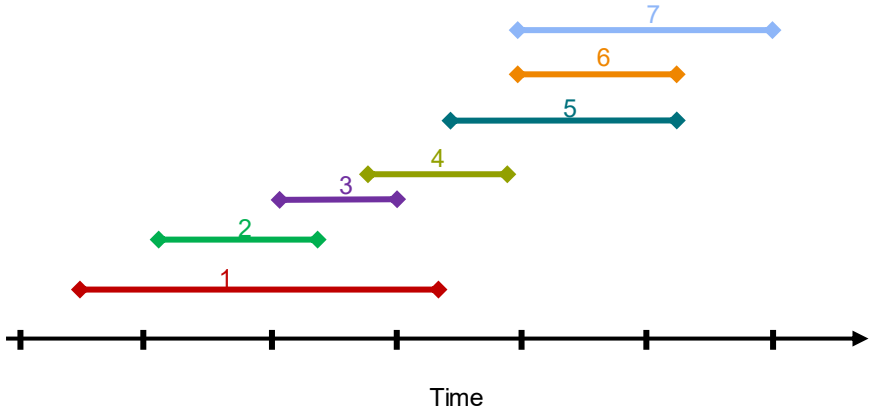

"Natural Order"

Start time: consider acts in ascending order of $s(i)$

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Shortest time: consider acts in ascending order of $f(i) - s(i)$

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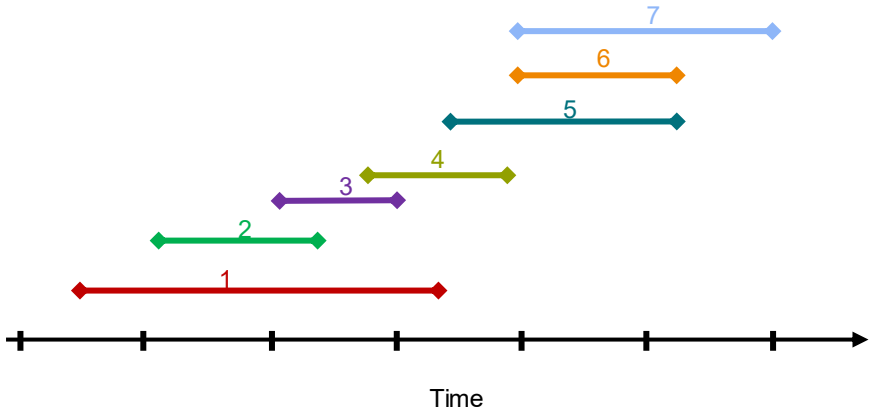
Fewest conflicts: sort in ascending order of $c(i)$, the number of acts overlapping with act i

- Optimal in this example



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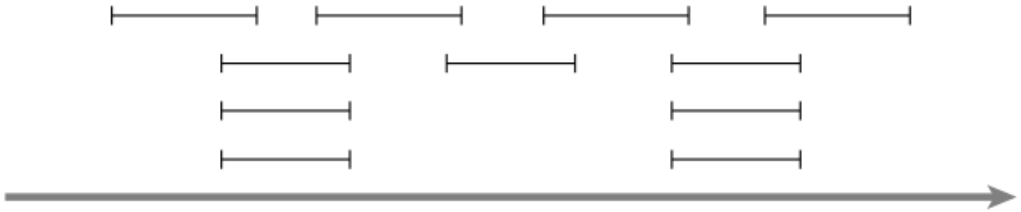
- But not in this one



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Shortest time: consider acts in ascending order of $f(i) - s(i)$

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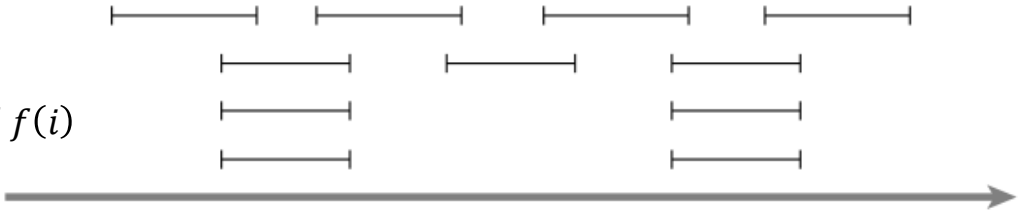
Finish time: consider acts in ascending order of $f(i)$

- We will show this is always optimal



Analysis

Let A be the set of acts returned by the algorithm when acts are sorted by finish time.
What do we need to prove?

- A is compatible (obvious property of algorithm)
- A is optimal

We will prove A is optimal by a "greedy stays ahead" argument

Greedy Stays Ahead

Let $A = i_1, \ldots, i_k$ be the intervals selected by our algorithm

Let $O = j_1, \ldots, j_m$ be the intervals of an optimal solution O

- There may be many optimal solutions
- Assume both are sorted by finish time

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Could it be the case that $m > k$?

- If A is an optimal solution, $m = k$

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Observation: $f(i_1) \leq f(j_1)$.

- The first show in A finishes no later than the first show in O

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Claim: $f(i_r) \leq f(j_r)$ for all $r = 1, 2, 3, \ldots$

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- Proof by induction on r
- Base case ($r = 1$): i_1 is the first choice of the greedy algorithm, which has the earliest overall finish time, so $f(i_1) \leq f(j_1)$

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We still need to show the inductive step

- Prove that if $f(i_{r-1}) \leq f(j_{r-1})$, then $f(i_r) \leq f(j_r)$

Inductive Step

Assume for induction that $f(i_{r-1}) \leq f(j_{r-1})$ for $r \geq 2$

- j_r is compatible with j_{r-1}, so $f(j_{r-1}) \leq s(j_r)$
- By inductive hypothesis, $f(i_{r-1}) \leq f(j_{r-1})$
- Thus, $f(i_{r-1}) \leq s(j_r)$ and interval j_r is in the set of available intervals when try to select i_r
- Since we greedily select the earliest finish time, $f(i_r) \leq f(j_r)$

Exercise II

Q: Recall that k is the number of intervals in the greedy solution and m is the number of intervals in an optimal solution. What have we just proven?

a) $f(i_r) \leq f(j_r)$ for all $r = 1, 2, \dots, m$

b) $f(i_r) \leq f(j_r)$ for all $r = 1, 2, \dots, k$

c) The greedy algorithm is optimal

d) None of the above

Exercise II

Q: Recall that k is the number of intervals in the greedy solution and m is the number of intervals in an optimal solution. What have we just proven?

a) $f(i_r) \leq f(j_r)$ for all $r = 1, 2, \dots, m$

b) $f(i_r) \leq f(j_r)$ for all $r = 1, 2, \dots, k$

c) The greedy algorithm is optimal

d) None of the above

Optimality

Can it be the case that $k < m$?

- For contradiction, suppose $k < m$
- Let's consider interval j_{k+1}
- We know that $f(i_k) \leq f(j_k) \leq s(j_{k+1})$
- So, interval j_{k+1} is compatible with greedy intervals A
- However, the greedy algorithm did not add j_{k+1} to A $(\rightarrow\leftarrow)$

Therefore, $k = m$

Running Time

$R \leftarrow$ set of all shows sorted by some property
$A \leftarrow \emptyset$
while R is not empty **do**
    take first show i from R
    add i to A
    delete i and all overlapping acts from R

Can we make the loop better than n^2?

Running Time

$R \leftarrow$ set of all shows sorted by some property
$A \leftarrow \emptyset$; end $\leftarrow 0$
for i from 1 to n **do**
    if $s(i) \geq$ end **then**
    add i to A
    end = $f(i)$

$\Theta(n \log n)$

- Running time dominated by sort
- Reduced loop to linear time swapping from while to for

Next Time

Greedy: make a single "greedy" choice at a time and don't look back

Focusing on proof techniques for correctness

- "Greedy Stays Ahead" (today)
- Exchange arguments (next time)