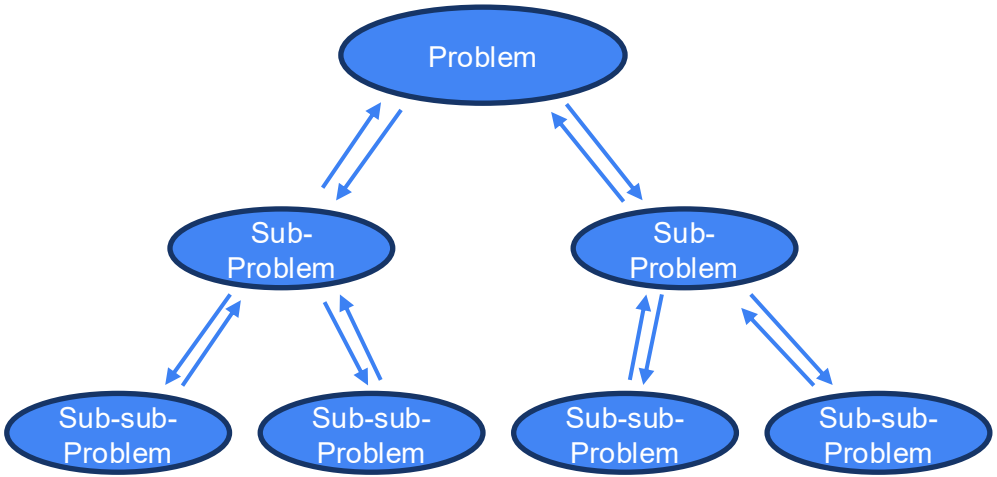
Lecture 15

Divide and Conquer I (take II)



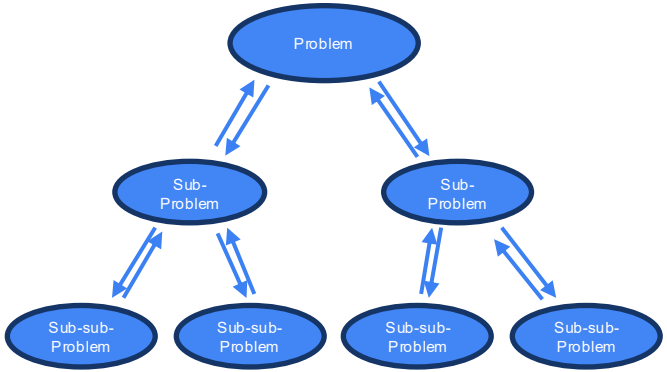
Announcements

- Homework 5 Reflection due this Sunday night (3/29)
- Group Meetings continue
- Individual Project 2 due next Sunday night (4/5)

Divide and Conquer Idea

General Recipe (or pattern)

- Divide problem into several parts
- Solve each part recursively
- Combine solutions to sub-problem into overall solution



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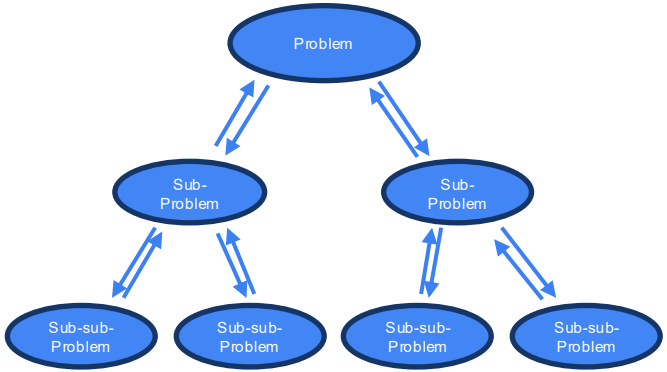
Our Focus

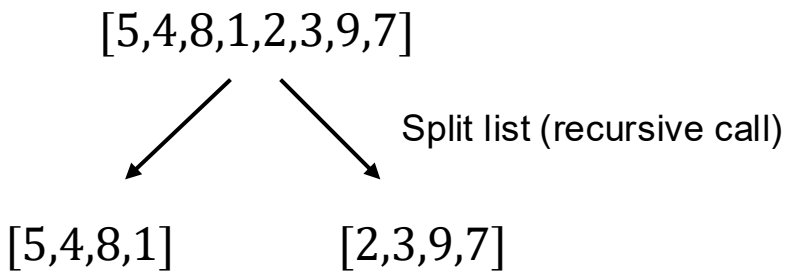
- Algorithm design
- Running time analysis



MergeSort Example

[5,4,8,1,2,3,9,7]

MergeSort Example



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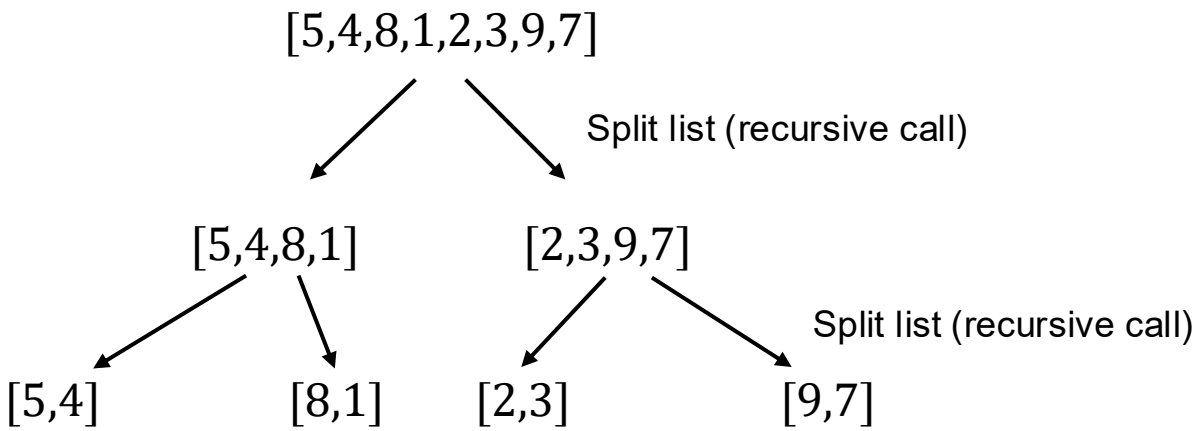

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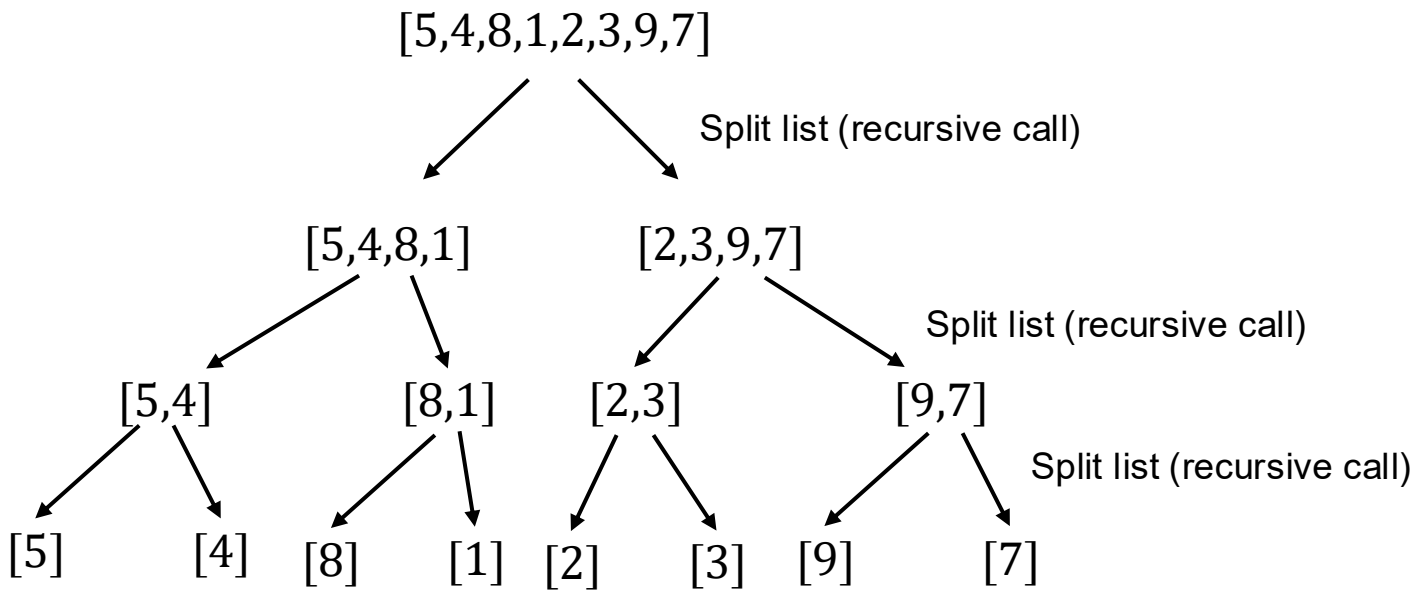
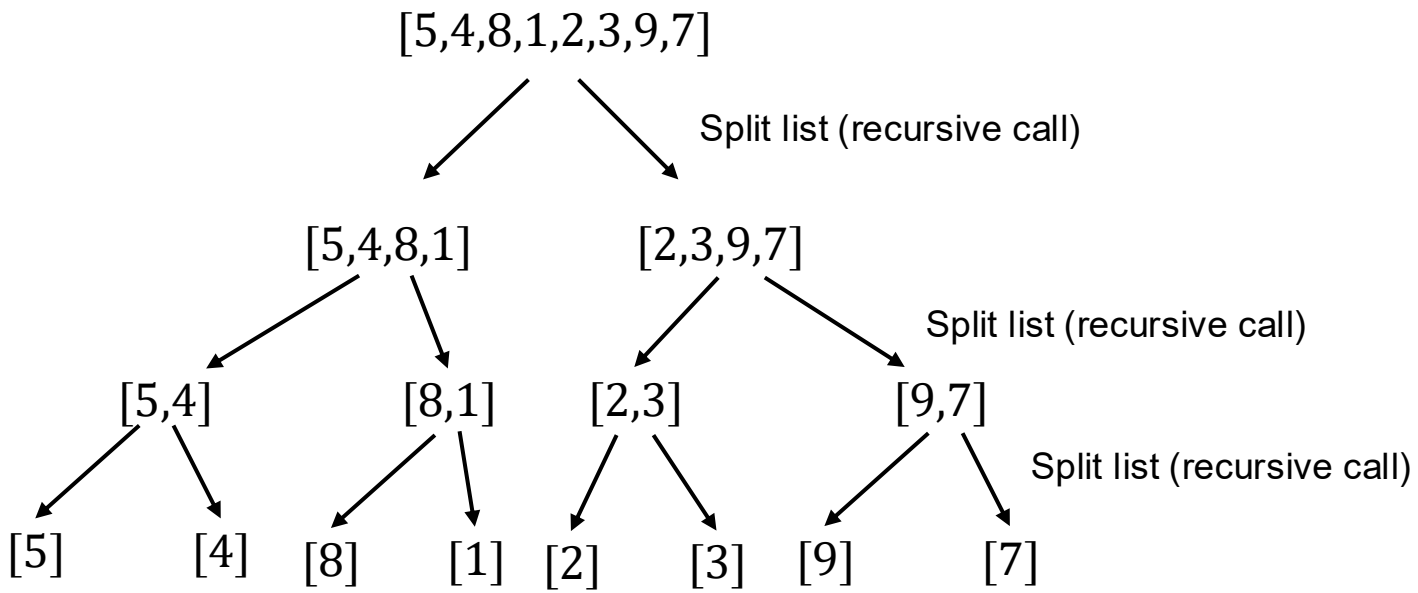


MergeSort Example



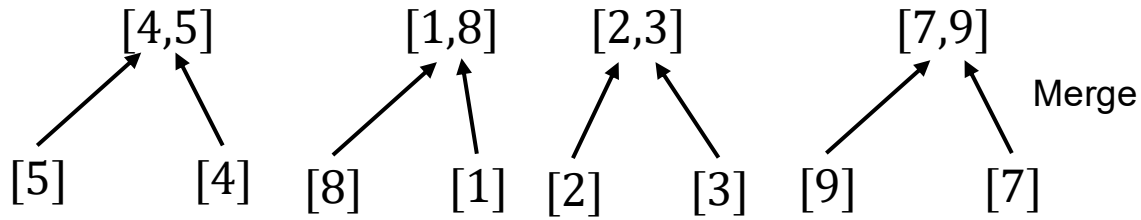
These lists are now sorted!

MergeSort Example

[5] [4] [8] [1] [2] [3] [9] [7]

Let’s recombine them

MergeSort Example



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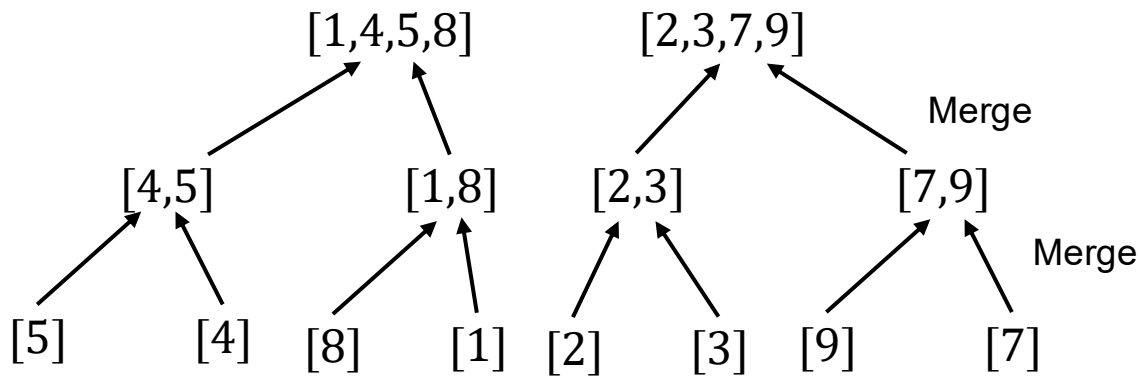

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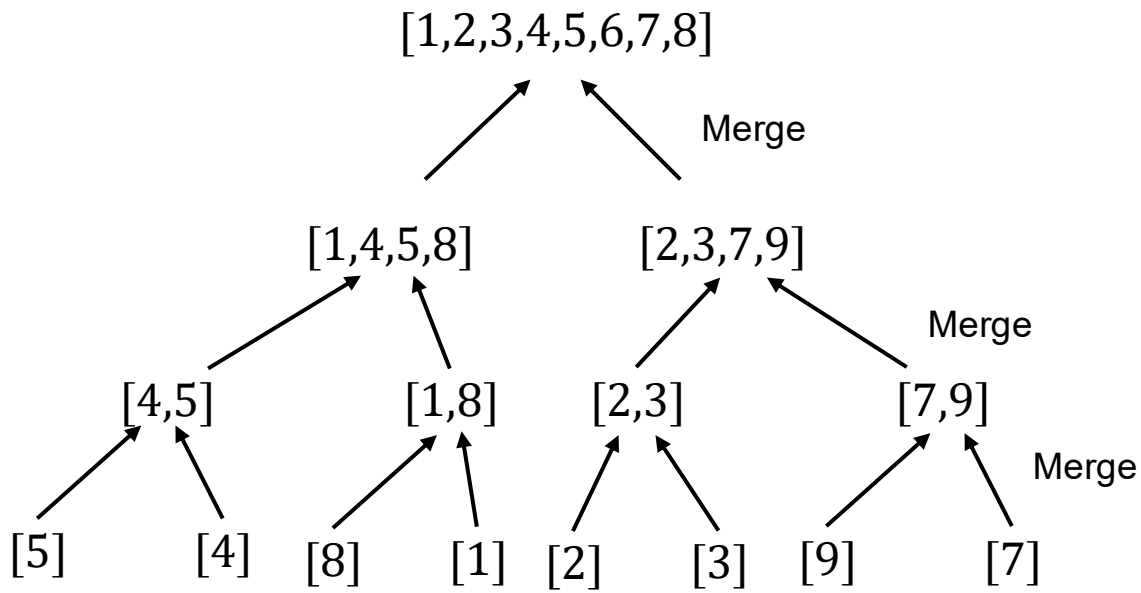

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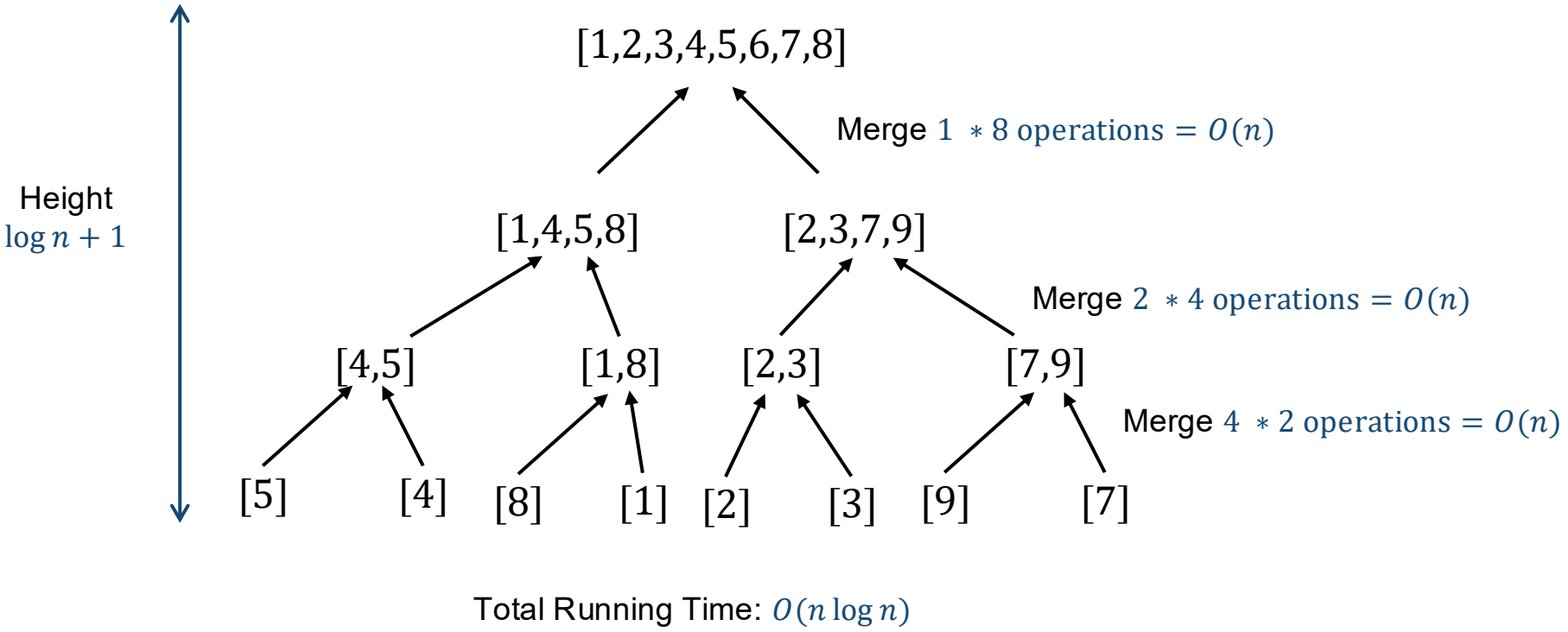

Total Running Time: $O(n \log n)$

MergeSort (Recursive Formulation)

Problem: given a list L of n unique items, arranger them in ascending order

$\text{MergeSort}(L)$:
    if L has one item **do**
        return L
    Divide L into two halves A, B
    $A \leftarrow \text{MergeSort}(\text{A})$
    $B \leftarrow \text{MergeSort}(\text{B})$
    $L \leftarrow \text{Merge}(A, B)$
    return L

MergeSort (Recursive Formulation)

Problem: given a list L of n unique items, arranger them in ascending order

Let $T(n)$ be the worst-case running time of $\mathrm{MergeSort(L)}$

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Base case: $O(1)$

Recursive cases: $T(n/2)$

Combination step: $\Theta(n)$

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$$T(n) = 2T\left({}^{n}/_{2}\right) + \Theta(n)$$

Recurrence Relations

A way of representing a function in terms of smaller values of the function

- Example: $T(n) = T(n-1) + 5; T(1) = 1$

  Recursive case Base case

- To be well defined, computing the recursive case must end in a base case

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$$T(3) = T(2) + 5 = T(1) + 2 * 5 = 1 + 2 * 5$$

- Extrapolating from this, we can write $T(n) = 1 + 5(n-1)$

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$$T(8) = T(4) + 5 = T(2) + 2 * 5 = T(1) + 3 * 5 = 1 + 3 * 5$$

- Extrapolating from this, we can write $T(n) = 1 + 5 \log n$

Recurrence

$T(n) = 2T(n/2) + \Theta(n)$

- $T(n)$ is defined in terms of small values of n
- For base case: $T(1)$ is $O(1)$

Goal: "solve" the recurrence by finding a simple expression for $T(n)$ for all n

Recurrence

Recurrence:

- $T(n) = 2T(n/2) + O(n)$
- $T(1) = O(1)$

Let's the use definition on Big-O:

- $T(n) \leq 2T(n/2) + cn$
- $T(1) \leq c$

Unrolling

Idea: "unroll" the recurrence

$$
\begin{aligned}
T(m) &\leq 2T(m/2) + cm && \text{definition of } T(n) \text{ for } n = m \\
&\leq 2[2T(m/4) + c(m/2)] + cm && \text{definition of } T(n) \text{ for } n = m/2 \\
&= 4T(m/4) + 2cm \\
&\leq 4[2T(m/8) + c(m/4)] + 2cm && \text{definition of } T(n) \text{ for } n = m/4 \\
&= 8T(m/4) + 3cm \\
&\leq \cdots \\
&\ldots \\
&= mT(m/m) + \log_2 m \cdot cm && \text{continue until we hit the base case} \\
&\leq mT(1) + \log_2 m \cdot cm \\
&\leq mc + \log_2 m \cdot cm && \text{definition of } T(1) \\
&= O(m \log m)
\end{aligned}
$$

Exercise 2

Q: suppose we have the recurrence $T(n) = T(n/2) + T(n/3)$. What do we get after two unrollings?

a. $T(n/4) + T(n/9)$

b. $T(n/4) + 2T(n/6) + T(n/9)$

c. $2T(n/6)$

d. $T(n/4) + T(n/6) + T(n/9)$

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Maximum Subsequence Sum

Input: a list L of n numbers

Goal: find the value of the largest subsequence sum (including empty subsequence)

Example

- $L = [4, -1, 8, -2, 3, -5, 1, -5]$

- MSS: 14 (sum of first five elements)

Exercise 3

Which of the following are true for a maximum sum subsequence?

a. It has more positive than negative numbers

b. It does not start or end with a negative number

c. Any maximal subsequence of negative numbers is bordered by a subsequence of positive numbers with a larger sum in absolute value

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Baseline MSS Algorithm

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- Idea: check all subsequences

MSS_baseline(L):
    $n \leftarrow len(L)$
    Initialize A as an $n \times n$ matrix of zeroes
    for $i = 1$ to n **do**
        sum $\leftarrow 0$
        for $j = i$ to n **do**
            sum += L[j]
            $A[i, j] \leftarrow$ sum
    return $\max_{i,j} A[i, j]$

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Can we do better?

Divide-and-Conquer for MSS

Recursive algorithm for MSS

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Recursive algorithm for MSS

Idea:

- Find MSS X in left half of list
- Find MSS Y in right half of list
- Find MSS Z for subsequences that cross the midpoint of list

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$$L = [4, -1, 8, -2, 3, -5, 1, -5]$$
$$X = 13;\ Y = 3;\ Z = 14$$

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$$L = [4, -1, 8, -2, 3, -5, 1, -5]$$
$$X = 13;\ Y = 3;\ Z = 14$$

- Return maximum of X, Y, Z
- How do we find X, Y, Z?

Divide-and-Conquer for MSS

$\text{MSS\_dc}(L, low, high)$:

if $high == low$ **do**

return $L[low]$

$mid \leftarrow \left\lfloor \frac{low+high}{2} \right\rfloor$

$X = \text{MSS\_dc}(L, low, mid)$

$Y = \text{MSS\_dc}(L, mid + 1, high)$

return $\max(X, Y, Z)$

Divide-and-Conquer for MSS

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$X = \text{MSS\_dc}(L, low, mid)$

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$sum \leftarrow 0; X' \leftarrow 0$

for $i = mid$ to low **do**

$sum \mathrel{+}= L[i]$

$X' = \max(sum, X')$

$sum \leftarrow 0; Y' \leftarrow 0$

for $j = mid + 1$ to $high$ **do**

$sum \mathrel{+}= L[j]$

$Y' = \max(sum, Y')$

$Z = X' + Y'$

return $\max(X, Y, Z)$

Divide-and-Conquer for MSS

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if $high == low$ **do**

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Base case: $O(1)$

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Recursive cases

$sum \leftarrow 0; X' \leftarrow 0$

for $i = mid$ to low **do**

$sum \mathrel{+}= L[i]$

$X' = \max(sum, X')$

Compute left part of Z

$sum \leftarrow 0; Y' \leftarrow 0$

for $j = mid + 1$ to $high$ **do**

$sum \mathrel{+}= L[j]$

$Y' = \max(sum, Y')$

Compute right part of Z

$Z = X' + Y'$

Compute Z

return $\max(X, Y, Z)$

Return max

Running Time

Let $T(n)$ be the running time of $\mathrm{MSS\_dc}(L)$ on a list of size n

- Base case is $O(1)$
- Two recursive calls on arrays of size $n/2$: $2T(n/2)$
- Work outside of recursive calls: $O(n)$

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Therefore, $T(n) = 2T(n/2) + O(n)$

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Therefore, $T(n) = 2T(n/2) + O(n)$

Implies the running time is $O(n \log n)$

Next Time

- Running time of more complex divide and conquer approaches